

Hylomorphic Vortices in Abelian Gauge Theories

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Abstract

We consider an Abelian Gauge Theory in \mathbb{R}^4 equipped with the Minkowski metric. This theory leads to a system of equations, the Klein-Gordon-Maxwell equations, which provide models for the interaction between the electromagnetic field and matter. We assume that the nonlinear term is such that the energy functional is positive; this fact makes the theory more suitable for physical models.

A three dimensional vortex is a finite energy, stationary solution of these equations such that the matter field has nontrivial angular momentum and the magnetic field looks like the field created by a finite solenoid. Under suitable assumptions, we prove the existence of three dimensional vortex-solutions.

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1 Introduction

Roughly speaking, a *vortex* is a *solitary wave* ψ with non-vanishing angular momentum ($\mathbf{M}(\psi) \neq 0$). A *solitary wave* is a solution of a field equation whose energy is localized and which preserves this localization in time.

Here we are interested in proving the existence of vortices in Abelian gauge theories. Abelian gauge theories, in \mathbb{R}^4 equipped with the Minkowski metric, provide models for the interaction between the electromagnetic field and matter. Actually an Abelian gauge theory leads to a system of equations (see (7), (8), (9)), the Klein-Gordon-Maxwell equations (KGM), which occur in various physical problems such as elementary particles, superconductivity, cosmology, nonlinear optics (see e.g. [26], [19], [24], [29]).

The KGM can be regarded as a perturbation of the nonlinear Klein-Gordon equation (KG) (see (3)).

So first we recall some existence results for KG:

- For the case $\mathbf{M}(\psi) = 0$, we recall the pioneering paper of Rosen [25] and [12], [27], [10]. When the lower order term $W \geq 0$ (see (3)), the spherically symmetric solitary waves have been called *Q*-balls by Coleman in [13] and this is the name used in the physics literature.
- Vortices for KG in two space dimensions have been investigated in [21]; later also three dimensional vortices for KG have been investigated (see [9], [2]).

Now let us see some literature on KGM. We notice that the peculiarities of the model depend on the lower order term W and it is relevant to distinguish various situations.

- For the case $\mathbf{M}(\psi) = 0$, the existence of solitary waves for KGM was first proved in [4] assuming that

$$W(s) = \frac{1}{2}s^2 - \frac{s^p}{p}, \quad 4 < p < 6, \quad s \geq 0. \quad (1)$$

The existence of solitary waves for KGM in this situation (i.e. with $\mathbf{M}(\psi) = 0$ and W as in (1)) has been studied also in [11], [14], [15], [16]. In these papers the existence and the non-existence of stationary solutions has been proved under different assumptions.

However the lower order term W defined by (1) is not suitable to model interesting physical models since in this case there are configurations with negative energy and the evolution problems relative to KGM does not possess in general global solutions (cf. e. g. [6]). So the request

$$W \geq 0$$

seems to be necessary to get solutions which are physically meaningful.

- The case $W \geq 0$ and $\mathbf{M}(\psi) = 0$ has been treated in [5].

Now let us consider the existence of vortices ($\mathbf{M}(\psi) \neq 0$) for KGM.

- The existence of vortices for Abelian gauge theories in two space dimensions has been discovered in a seminal paper by Abrikosov [1] in the study of the superconductivity. Then, in [23], the planar vortices are studied in the context of elementary particles (see also the books [19], [24], [26], [30] with their references). We point out that, in these cases, the function W that has been considered is of the type

$$W(s) = (1 - s^2)^2 \tag{2}$$

namely it is a double well shaped and positive function. However, if (2) holds, there are not vortices in 3 space dimensions for KGM (see Theorem 2).

- In [8], [7] the existence of vortices in 3 space dimensions has been proved assuming (1).

The aim of this paper is to prove the existence of vortices in 3 space dimensions also when $W(s) \geq 0$ and $W(0) = 0$. More precisely we consider the term W with the assumption used in [13] and similar papers.

The existence of solitary waves in this situation is based on the fact that the ratio between energy and charge can be sufficiently low; thus, following [3], these vortices are called hylomorphic (cf. section 2.2).

Moreover these vortices are related with a non trivial magnetic field and a non trivial electric field. In particular the magnetic field looks like to the field created by a finite solenoid.

Since the KGM are invariant for the Lorentz group, a Lorentz boost of a vortex creates a travelling solitary wave.

The paper is organized as follows. In section 2 we introduce the KGM-equations, we study some of their general features, we give the definition of three dimensional vortex and finally state the main result in Theorem 3. Section 3 is devoted to the proof of Theorem 3.

2 Statement of the problem and results

2.1 The Klein-Gordon-Maxwell system

The nonlinear Klein-Gordon equation for a complex valued field ψ , defined on the spacetime \mathbb{R}^4 , can be written as follows:

$$\square\psi + W'(|\psi|)\frac{\psi}{|\psi|} = 0 \quad (3)$$

where

$$\square\psi = \frac{\partial^2\psi}{\partial t^2} - \Delta\psi, \quad \Delta\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

and $W : \mathbb{R}_+ \rightarrow \mathbb{R}$.

Hereafter $x = (x_1, x_2, x_3)$ and t will denote the space and time variables.

The field $\psi : \mathbb{R}^4 \rightarrow \mathbb{C}$ will be called *matter field*. If $W'(s)$ is linear, $W'(s) = m_0^2 s$, $m_0 \neq 0$, equation (3) reduces to the Klein-Gordon equation.

Now let Γ be a 1- form on \mathbb{R}^4 whose coefficients Γ_j are in the Lie algebra $u(1)$ of the group $U(1) = S^1$, i.e. $\Gamma_j = -iA_j$, where i is the imaginary unit and A_j ($j = 0, \dots, 3$) are real maps defined in \mathbb{R}^4 .

Consider the Abelian gauge theory related to ψ and to Γ and described by the Lagrangian density (see e.g. [30], [26])

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 - W(|\psi|) \quad (4)$$

where

$$\mathcal{L}_0 = -\frac{1}{2} \langle d_A \psi, d_A \psi \rangle, \quad \mathcal{L}_1 = -\frac{1}{2} \langle d_A A, d_A A \rangle, \quad A = \sum_{j=0}^3 A_j dx^j$$

and W is a real C^1 -function. Here

$$d_A = d - iqA = \sum_{j=0}^3 \left(\frac{\partial}{\partial x^j} - iqA_j \right)$$

denotes the gauge covariant differential and $\langle \cdot, \cdot \rangle$ denotes the scalar product between forms with respect the Minkowski metric in \mathbb{R}^4 and q is a constant.

Since the A_j 's are real,

$$d_A A = dA - iA \wedge A = dA.$$

From now on, we shall use the following notation:

$$\mathbf{A} = (A_1, A_2, A_3) \text{ and } \phi = -A_0.$$

If we set $t = x^0 = -x_0$ and $x = (x^1, x^2, x^3) = (x_1, x_2, x_3)$ and use vector notation, the Lagrangian densities $\mathcal{L}_0, \mathcal{L}_1$ can be written as follows

$$\mathcal{L}_0 = \frac{1}{2} \left[|(\partial_t + iq\phi)\psi|^2 - |(\nabla - iq\mathbf{A})\psi|^2 \right]. \quad (5)$$

$$\mathcal{L}_1 = \frac{1}{2} |\partial_t \mathbf{A} + \nabla \phi|^2 - \frac{1}{2} |\nabla \times \mathbf{A}|^2.$$

Here $\nabla \times$ and ∇ denote respectively the curl and the gradient operators with respect to the x variable.

Now consider the total action of the Abelian gauge theory

$$\mathcal{S} = \int (\mathcal{L}_0 + \mathcal{L}_1 - W(|\psi|)) dx dt. \quad (6)$$

Making the variation of \mathcal{S} with respect to ψ , ϕ and \mathbf{A} we get the system of equations (KGM)

$$(\partial_t + iq\phi)^2 \psi - (\nabla - iq\mathbf{A})^2 \psi + W'(|\psi|) \frac{\psi}{|\psi|} = 0 \quad (7)$$

$$\nabla \cdot (\partial_t \mathbf{A} + \nabla \phi) = q \left(\text{Im} \frac{\partial_t \psi}{\psi} + q\phi \right) |\psi|^2 \quad (8)$$

$$\nabla \times (\nabla \times \mathbf{A}) + \partial_t (\partial_t \mathbf{A} + \nabla \phi) = q \left(\text{Im} \frac{\nabla \psi}{\psi} - q\mathbf{A} \right) |\psi|^2. \quad (9)$$

Here $\nabla \cdot$ denotes the divergence operator.

In order to show the relation of the above equations with the Maxwell equations and to get a model for Electrodynamics, we make the following change of variables:

$$\mathbf{E} = - \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \phi \right) \quad (10)$$

$$\mathbf{H} = \nabla \times \mathbf{A} \quad (11)$$

$$\rho = -q \left(\operatorname{Im} \frac{\partial_t \psi}{\psi} + q\phi \right) |\psi|^2 \quad (12)$$

$$\mathbf{j} = q \left(\operatorname{Im} \frac{\nabla \psi}{\psi} - q\mathbf{A} \right) |\psi|^2. \quad (13)$$

So (8) and (9) are the second couple of the Maxwell equations with respect to a matter distribution whose electric charge and current densities are respectively ρ and \mathbf{j} :

$$\nabla \cdot \mathbf{E} = \rho \quad (14)$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}. \quad (15)$$

Equations (10) and (11) give rise to the first couple of the Maxwell equation:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = 0 \quad (16)$$

$$\nabla \cdot \mathbf{H} = 0. \quad (17)$$

If we set

$$\psi(t, x) = u(t, x) e^{iS(t, x)}, \quad u \in \mathbb{R}^+, \quad S \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}$$

equation (7) can be splitted in the two following ones

$$\square u + W'(u) + \left[|\nabla S - q\mathbf{A}|^2 - \left(\frac{\partial S}{\partial t} + q\phi \right)^2 \right] u = 0 \quad (18)$$

$$\frac{\partial}{\partial t} \left[\left(\frac{\partial S}{\partial t} + q\phi \right) u^2 \right] - \nabla \cdot [(\nabla S - q\mathbf{A}) u^2] = 0 \quad (19)$$

and these equations, using the variables \mathbf{j} and ρ become

$$\square u + W'(u) + \frac{\mathbf{j}^2 - \rho^2}{q^2 u^3} = 0 \quad (20)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \quad (21)$$

Equation (21) is the charge continuity equation.

Notice that equation (21) is a consequence of (14) and (15).

In conclusion, an Abelian gauge theory, via equations (20,14,15,16,17), provides a model of interaction of the matter field ψ with the electromagnetic field (\mathbf{E}, \mathbf{H}) .

2.2 The Hamilton-Jacobi equations

Observe that the Lagrangian (4) is invariant with respect to the gauge transformations

$$\psi \rightarrow e^{iq\chi}\psi \quad (22)$$

$$\phi \rightarrow \phi - \partial_t\chi \quad (23)$$

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi \quad (24)$$

where $\chi \in C^\infty(\mathbb{R}^4)$.

So, our equations are gauge invariant; if we use the variable $u, \rho, \mathbf{j}, \mathbf{E}, \mathbf{H}$, this fact can be checked directly since these variables are gauge invariant.

In fact, equations (14,15,16,17,20) are the gauge invariant formulation of equations (7,8,9).

Also, we can replace the variables ρ, \mathbf{j} with the variables Ω and \mathbf{K} defined by the following equations:

$$\rho = q\Omega u^2 \quad (25)$$

$$\mathbf{j} = q\mathbf{K}u^2. \quad (26)$$

Using this notation, the continuity equation (21) becomes

$$\partial_t(\Omega u^2) + \nabla \cdot (\mathbf{K}u^2) = 0.$$

This equation allows us to interpret the matter field to be a fluid composed by particles whose density is given by Ωu^2 and which move in a velocity field

$$\mathbf{v} = \frac{\mathbf{K}}{\Omega} = -\frac{\nabla S - q\mathbf{A}}{\partial_t S + q\phi}. \quad (27)$$

Then, since ρ represents the electric charge density, $q = \rho/\Omega u^2$ is interpreted as the electric charge of each particle. The total number of particles

$$\sigma = \int \Omega u^2 dx = - \int (\partial_t S + q\phi) u^2 dx \quad (28)$$

is an integral of motion which, following [3], we will call *hylenic charge*.

We set

$$W(s) = \frac{m^2}{2}s^2 + N(s), \quad (29)$$

with $N(0) = N'(0) = N''(0) = 0$; then Equation (20), using the variables Ω and \mathbf{K} , becomes

$$\square u + N'(u) + (m^2 + \mathbf{K}^2 - \Omega^2)u = 0 \quad (30)$$

If

$$\square u + N'(u) \ll u, \quad (31)$$

this equation, can be approximated by

$$\Omega^2 = m^2 + \mathbf{K}^2;$$

and, using the definition of Ω and \mathbf{K} we get

$$(\partial_t S + q\phi)^2 = m^2 + (\nabla S - q\mathbf{A})^2$$

or

$$\partial_t S = -q\phi + \sqrt{m^2 + (\nabla S - q\mathbf{A})^2}. \quad (32)$$

This is the relativistic Hamilton-Jacobi equation of a particle of rest mass m and charge q in a electromagnetic field with gauge potentials (ϕ, \mathbf{A}) (cf. e.g. [22] Ch. III). Equations (32) and (27) completely describe the motion of these particles.

Since S is a phase, then

$$\begin{aligned} \partial_t S &= \omega \\ \nabla S &= \mathbf{k}; \end{aligned}$$

where ω and \mathbf{k} are the local frequency and the local wave number repectively. Moreover, the energy of each particle moving according to (32), is given by

$$E = \partial_t S$$

and its momentum is given by

$$\mathbf{p} = \nabla S;$$

thus we have that

$$\begin{aligned} E &= \omega \\ \mathbf{p} &= \mathbf{k}; \end{aligned}$$

these two equations are the De Broglie relation with $\hbar = 1$; it is interesting to see how the De Broglie relations arise in a natural way also out of quantum mechanics. We notice that in the De Broglie interpretation of quantum mechanics, Ω and \mathbf{K} represent the probability density and the probability flow of the position of a particle (see [17]).

If we do not assume (31), equation (32) needs to be replaced by

$$\partial_t S = -q\phi + \sqrt{m^2 + (\nabla S - q\mathbf{A})^2 + \frac{\square u + N'(u)}{u}}. \quad (33)$$

Concluding, we may think that equation (7) describes a fluid of particles of mass m and charge q which moves under the action of an electromagnetic field (\mathbf{E}, \mathbf{H}) ; the term $\frac{\square u + N'(u)}{u}$ in (33) can be regarded as a field describing a sort of interaction between particles. In the Bohm-De Broglie formulation of quantum mechanics, this term corresponds to the *quantum potential*.

2.3 Conservation laws

Noether's theorem states that any invariance for a one-parameter group of the Lagrangian implies the existence of an integral of motion (see e.g. [20]).

In the previous section, we have seen that the hylenic charge σ and, consequently, the electric charge $Q = q\sigma$ are integrals of motions. This conservation law is due to the gauge invariance.

Now we will consider other integrals which will be relevant for this paper.

- **Energy.** Energy, by definition, is the quantity which is preserved by the time invariance of the Lagrangian; using the gauge invariant variables, it takes the following form

$$\mathcal{E} = \mathcal{E}_m + \mathcal{E}_f \quad (34)$$

where

$$\mathcal{E}_m = \frac{1}{2} \int \left[\left(\frac{\partial u}{\partial t} \right)^2 + |\nabla u|^2 + (m^2 + \Omega^2 + \mathbf{K}^2)u^2 \right] + \int N(u)$$

and

$$\mathcal{E}_f = \frac{1}{2} \int (\mathbf{E}^2 + \mathbf{H}^2) dx.$$

(for the computation of \mathcal{E} , see e.g. ([6])).

- **Momentum.** Momentum, by definition, is the quantity which is preserved by the space invariance of the Lagrangian; using the gauge invariant variables, it takes the following form

$$\mathbf{P} = \mathbf{P}_m + \mathbf{P}_f \quad (35)$$

where

$$\mathbf{P}_m = \int [-(\partial_t u \nabla u dx) + \mathbf{K} \Omega u^2] dx$$

and

$$\mathbf{P}_f = \int \mathbf{E} \times \mathbf{H} \, dx.$$

- **Angular momentum.** The angular momentum, by definition, is the quantity which is preserved by virtue of the invariance under space rotations of the Lagrangian with respect to the origin. Using the gauge invariant variables, we get:

$$\mathbf{M} = \mathbf{M}_m + \mathbf{M}_f \quad (36)$$

where

$$\mathbf{M}_m = \int [-\mathbf{x} \times (\nabla u \, \partial_t u) + \mathbf{x} \times \mathbf{K} \Omega u^2] \, dx \quad (37)$$

and

$$\mathbf{M}_f = \int \mathbf{x} \times (\mathbf{E} \times \mathbf{H}) \, dx.$$

Notice that each of the integrals \mathcal{E} , \mathbf{P} , \mathbf{M} can be splitted in two parts (see (34), (35), (36)). The first one refers to the "matter field" and the second to the "electromagnetic field".

2.4 Stationary solutions and vortices

We look for stationary solutions of (7), (8), (9), namely solutions of the form

$$\psi(t, x) = u(x) e^{iS(x,t)}, \quad u \in \mathbb{R}^+, \quad \omega \in \mathbb{R}, \quad S = S_0(x) - \omega t \in \frac{\mathbb{R}}{2\pi\mathbb{Z}} \quad (38)$$

$$\partial_t \mathbf{A} = 0, \quad \partial_t \phi = 0. \quad (39)$$

Substituting (38) and (39) in (7), (8), (9), we get the following equations:

$$-\Delta u + \left[|\nabla S_0 - q\mathbf{A}|^2 - (\omega - q\phi)^2 \right] u + W'(u) = 0 \quad (40)$$

$$-\nabla \cdot [(\nabla S_0 - q\mathbf{A}) u^2] = 0 \quad (41)$$

$$-\Delta \phi = q(\omega - q\phi) u^2 \quad (42)$$

$$\nabla \times (\nabla \times \mathbf{A}) = q(\nabla S_0 - q\mathbf{A}) u^2. \quad (43)$$

Observe that equation (41) easily follows from equation (43). Then we are reduced to study the system (40), (42), (43). The energy of a solution of

equations (40), (42), (43) has the following expression

$$\begin{aligned} \mathcal{E} = & \frac{1}{2} \int \left(|\nabla u|^2 + |\nabla \phi|^2 + |\nabla \times \mathbf{A}|^2 + (|\nabla S_0 - q\mathbf{A}|^2 + (\omega - q\phi)^2) u^2 \right) \\ & + \int W(u) \end{aligned} \quad (44)$$

Moreover the (electric) charge (see (25) and (28)) is given by

$$Q = q\sigma \quad (45)$$

where

$$\sigma = \int \Omega u^2 = \int (\omega - q\phi) u^2. \quad (46)$$

Clearly, when $u = 0$, the only finite energy gauge potentials which solve (42), (43) are the trivial ones $\mathbf{A} = 0$, $\phi = 0$.

It is possible to have three types of finite energy stationary non trivial solutions:

- electrostatic solutions: $\mathbf{A} = 0$, $\phi \neq 0$;
- magnetostatic solutions: $\mathbf{A} \neq 0$, $\phi = 0$;
- electro-magneto-static solutions: $\mathbf{A} \neq 0$, $\phi \neq 0$.

Under suitable assumptions, all these types of solutions exist. The existence and the non existence of electrostatic solutions for the equations (40), (42) have been proved under different assumptions on W . In [4], [11], [14], [15], [16] lower order terms W like (1) have been taken into account. In [5] the existence of electrostatic solutions has been proved for a class of positive lower order terms W . In particular the existence of radially symmetric, electrostatic solutions has been analyzed. These solutions have zero angular momentum.

Here we are interested in electro-magneto-static solutions, in particular we shall study the existence of vortices which are solutions with nonvanishing angular momentum.

We set

$$\Sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0\}$$

and we define the map

$$\begin{aligned} \theta : \mathbb{R}^3 \setminus \Sigma &\rightarrow \frac{\mathbb{R}}{2\pi\mathbb{Z}} \\ \theta(x_1, x_2, x_3) &= \text{Im} \log(x_1 + ix_2). \end{aligned}$$

In (38) we take $S_0 = \ell\theta$ (ℓ integer) and give the following definition.

Definition 1 *A finite energy solution of Eq. (40), (42), (43) is called vortex if $S_0 = \ell\theta(x)$ with $\ell \neq 0$.*

In this case, ψ has the following form

$$\psi(t, x) = u(x) e^{i(\ell\theta(x) - \omega t)}; \quad \ell \in \mathbb{Z} - \{0\}. \quad (47)$$

We shall see (Proposition 7) that the angular momentum \mathbf{M}_m of the matter field of a vortex does not vanish; this fact justifies the name "vortex".

Observe that $\theta \in C^\infty(\mathbb{R}^3 \setminus \Sigma, \frac{\mathbb{R}}{2\pi\mathbb{Z}})$. We set with abuse of notation

$$\nabla\theta(x) = \frac{x_2}{x_1^2 + x_2^2} \mathbf{e}_1 - \frac{x_1}{x_1^2 + x_2^2} \mathbf{e}_2$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the canonical base in \mathbb{R}^3 .

Using the ansatz (47), equations (40), (42), (43) become

$$-\Delta u + \left[|\ell\nabla\theta - q\mathbf{A}|^2 - (\omega - q\phi)^2 \right] u + W'(u) = 0 \quad (48)$$

$$-\Delta\phi = q(\omega - q\phi)u^2 \quad (49)$$

$$\nabla \times (\nabla \times \mathbf{A}) = q(\ell\nabla\theta - q\mathbf{A})u^2. \quad (50)$$

In this case, the gauge invariant variables take the following expression:

$$\mathbf{E} = -\nabla\phi \quad (51)$$

$$\mathbf{H} = \nabla \times \mathbf{A} \quad (52)$$

$$\Omega = \omega - q\phi \quad (53)$$

$$\mathbf{K} = \ell\nabla\theta - q\mathbf{A} \quad (54)$$

$$\rho = q\Omega u^2 \quad (55)$$

$$\mathbf{j} = q\mathbf{K}u^2. \quad (56)$$

which give the equations

$$\Delta u - W'(u) = (\mathbf{K}^2 - \Omega^2) u$$

$$\nabla \cdot \mathbf{E} = q\Omega u^2$$

$$\nabla \times \mathbf{H} = q\mathbf{K}u^2.$$

As we have observed in the introduction, positive, double well shaped potentials W like (2) are not suitable for the existence of 3-dimensional vortices of type (47). In fact the following proposition holds:

Proposition 2 *Assume that W satisfies the assumptions:*

$$\forall s \geq 0 : W(s) \geq 0 \quad (57)$$

$$W(0) > 0 \quad (58)$$

$$\exists \bar{s} : W(\bar{s}) = 0 \quad (59)$$

then (40), (42), (43) has no vortex solution..

Proof. We shall prove that any configuration of the type

$$(u(x) e^{i(\ell\theta - \omega t)}, \phi, \mathbf{A}), \ell \in \mathbb{Z}, \ell \neq 0 \quad (60)$$

has infinite energy \mathcal{E} (44). Arguing by contradiction assume that (60) has finite energy. Since W satisfies (57) (58) and (59), the finiteness of the energy implies

$$\int W(u) < \infty$$

so that

$$u(\infty) = \bar{s}.$$

So, using again the finiteness of \mathcal{E} , we get

$$\int |\ell \nabla \theta - q \mathbf{A}|^2 < \infty.$$

So, if we take $0 < \delta_1 < \delta_2$, for all $\varepsilon > 0$ there exists $M > 0$ s.t. for all $x = (x_1, x_2, x_3)$ with

$$\delta_1 < r < \delta_2, \quad |x_3| > M, \quad r = \sqrt{x_1^2 + x_2^2},$$

we have

$$|\ell \nabla \theta - q \mathbf{A}| < \varepsilon.$$

So, for such (x_1, x_2, x_3) , we get

$$\frac{|\ell|}{\delta_2} - \varepsilon < \frac{|\ell|}{r} - \varepsilon = |\ell \nabla \theta| - \varepsilon < |\mathbf{A}(x)|.$$

Then, if ε is small enough, we get

$$0 < \mu = \frac{|\ell|}{\delta_2} - \varepsilon < |\mathbf{A}(x)|.$$

So

$$\infty = \int_{\delta_1}^{\delta_2} r dr \int_{|x_3| > M} \mu^6 dx_3 \leq \int |\mathbf{A}(x)|^6 dx.$$

Therefore $\mathbf{A} \notin L^6(\mathbb{R}^3)$ and then, by Sobolev inequality,

$$\int |\nabla \mathbf{A}|^2 dx = \infty.$$

This contradicts the finiteness of the energy \mathcal{E} . ■

2.5 The main existence result

Let W satisfy the following assumptions:

- W1) $\forall s \geq 0 : W(s) \geq 0$
- W2) W is C^2 with $W(0) = W'(0) = 0$, $W''(0) = m^2 > 0$,
- W3) $\inf_{s>0} \left(\frac{W(s)}{\frac{m^2}{2}s^2} \right) < 1$

We shall set

$$W(s) = \frac{m^2}{2}s^2 + N(s).$$

Clearly assumption W3) is equivalent to require that there exists $s_0 > 0$ such that

$$N(s_0) < 0. \tag{61}$$

By rescaling time and space we can assume without loss of generality

$$m^2 = 1.$$

Moreover, for technical reasons it is useful to assume that W is defined for all $s \in \mathbb{R}$ just setting

$$W(s) = W(-s) \text{ for } s < 0.$$

Now we can state the main existence result.

Theorem 3 *Assume that the function W satisfies assumptions W1), W2), W3). Then for all $\ell \in \mathbb{Z}$ there exists $\bar{q} > 0$ such that for every $0 \leq q \leq \bar{q}$ the*

equations (48), (49), (50) admit a finite energy solution in the sense of distributions $(u, \omega, \phi, \mathbf{A})$, $u \neq 0$, $\omega > 0$. The maps u , ϕ depend only on the variables $r = \sqrt{x_1^2 + x_2^2}$ and x_3

$$u = u(r, x_3), \quad \phi = \phi(r, x_3).$$

and the magnetic potential \mathbf{A} has the following form

$$\mathbf{A} = a(r, x_3) \nabla \theta = a(r, x_3) \left(\frac{x_2}{r^2} \mathbf{e}_1 - \frac{x_1}{r^2} \mathbf{e}_2 \right) \quad (62)$$

If $q = 0$, then $\phi = 0$, $\mathbf{A} = 0$. If $q > 0$ then $\phi \neq 0$. Moreover $\mathbf{A} \neq 0$ if and only if $\ell \neq 0$

Remark 4 When there is no coupling with the electromagnetic field, i.e. $q = 0$, equations (48), (49), (50) reduce to find vortices to the nonlinear Klein-Gordon equation and an analogous result has been obtained in [2].

Remark 5 When $\ell = 0$ and $q > 0$ the last part of Theorem 3 states the existence of electrostatic solutions, namely finite energy solutions with $u \neq 0$, $\phi \neq 0$ and $\mathbf{A} = 0$. This result generalizes a recent theorem (see [5]), where the existence of electrostatic solutions has been stated under assumptions stronger than $W1, W2, W3$.

Remark 6 By the presence of the term $\nabla \theta$ equations (48), (50) are not invariant under the $O(3)$ group action as it happens for the equations (7), (8), (9) we started from. Indeed there is a breaking of radial symmetry and the solutions u , ϕ , \mathbf{A} in theorem 3 have only an S^1 (cylindrical) symmetry.

Proposition 7 Let $(u, \omega, \phi, \mathbf{A})$ be a non trivial, finite energy solution of equations (48), (49), (50) as in theorem 3. Then the angular momentum \mathbf{M}_m (see (37)) has the following expression

$$\mathbf{M}_m = - \left[\int (\ell - qa) (\omega - q\phi) u^2 dx \right] \mathbf{e}_3 \quad (63)$$

and, if $\ell \neq 0$, it does not vanish.

Proof. By (54) and (62), we have that

$$\mathbf{K} = \nabla S - q\mathbf{A} = \ell \nabla \theta - qa \nabla \theta = (\ell - qa) \nabla \theta. \quad (64)$$

Then, using (53) and (64), we have that

$$\mathbf{M}_m = \int \mathbf{x} \times \mathbf{K} \Omega u^2 dx = \int \mathbf{x} \times \nabla \theta (\ell - qa) (\omega - q\phi) u^2 dx.$$

Let us compute

$$\begin{aligned}
\mathbf{x} \times \nabla \theta &= (x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3) \times \left(\frac{x_2}{r^2} \mathbf{e}_1 - \frac{x_1}{r^2} \mathbf{e}_2 \right) \\
&= -\frac{x_1^2}{r^2} \mathbf{e}_3 - \frac{x_2^2}{r^2} \mathbf{e}_3 + \frac{x_2 x_3}{r^2} \mathbf{e}_2 + \frac{x_1 x_3}{r^2} \mathbf{e}_1 \\
&= \frac{x_1 x_3}{r^2} \mathbf{e}_1 + \frac{x_2 x_3}{r^2} \mathbf{e}_2 - \mathbf{e}_3.
\end{aligned}$$

Then

$$\mathbf{M}_m(\psi) = \int \left(\frac{x_1 x_3}{r^2} \mathbf{e}_1 + \frac{x_2 x_3}{r^2} \mathbf{e}_2 - \mathbf{e}_3 \right) (\ell - qa) (\omega - q\phi) u^2 dx. \quad (65)$$

On the other hand, since the functions $x_1 x_3 \frac{(\ell - qa)(\omega - q\phi)u^2}{r^2}$ and $x_2 x_3 \frac{(\ell - qa)(\omega - q\phi)u^2}{r^2}$ are odd in x_1 and x_2 respectively, we have

$$\int x_1 x_3 \frac{(\ell - qa)(\omega - q\phi)u^2}{r^2} = \int x_2 x_3 \frac{(\ell - qa)(\omega - q\phi)u^2}{r^2} = 0. \quad (66)$$

Then (63) follows from (65) and (66). Now let $\ell \neq 0$. In order to see that $\mathbf{M}_m \neq 0$, it is sufficient to prove that

$$(\ell - qa)(\omega - q\phi) > 0. \quad (67)$$

or that

$$(\ell - qa)(\omega - q\phi) < 0. \quad (68)$$

Clearly, since $\ell, \omega \neq 0$ (67) or (68) are satisfied when $q = 0$. Now let $q > 0$. Assume that $\ell > 0$ and we show that (67) is verified. The case $\ell < 0$ can be treated analogously.

By (42) we have that

$$-\Delta \phi + q^2 u^2 \phi = q \omega u^2.$$

Since ω/q is a supersolution, by the maximum principle, $\phi < \omega/q$ and hence $\omega - q\phi > 0$. So, in order to prove (67), it remains to show that

$$\ell - qa > 0 \quad (69)$$

By (43) we have that

$$\nabla \times (\nabla \times \mathbf{A}) = q(\ell \nabla \theta - q \mathbf{A}) u^2. \quad (70)$$

Now a straight computation shows that,

$$\nabla \times (\nabla \times a \nabla \theta) = b \nabla \theta \quad (71)$$

where

$$b = -\frac{\partial^2 a}{\partial r^2} + \frac{1}{2} \frac{\partial a}{\partial r} - \frac{\partial^2 a}{\partial x_3^2}.$$

Then, setting $\mathbf{A} = a \nabla \theta$ in (70) and using (71), we have

$$-\frac{\partial^2 a}{\partial r^2} + \frac{1}{2} \frac{\partial a}{\partial r} - \frac{\partial^2 a}{\partial x_3^2} = q(\ell - qa) u^2.$$

Since ℓ/q is a supersolution, by the maximum principle, $a < \ell/q$ and hence (69) is proved. ■

Remark 8 *Observe that in the interpretation given in 2.2, the quantity $\Omega u^2 = (\omega - q\phi) u^2$ represents the density of particles; then by (63) $-(\ell - qa) \mathbf{e}_3$ represents the angular momentum of each particle. So, since ℓ is an integer, we see that the classical model described by this Abelian gauge theory presents a quantization phenomenon. Notice also that, for $q = 0$, the angular momentum of each particle takes only integer values.*

Finally let us observe that under general assumptions on W , magneto-static solutions (i.e. with $\omega = \phi = 0$) do not exist. In fact the following proposition holds:

Proposition 9 *Assume that W satisfies the assumptions $W(0) = 0$ and $W'(s)s \geq 0$. Then (48), (49), (50) has no solutions with $\omega = \phi = 0$.*

Proof. Set $\omega = 0$, $\phi = 0$ in (48) and we get

$$-\Delta u + |\ell \nabla \theta - q \mathbf{A}|^2 u + W'(u) = 0.$$

Then, multiplying by u and integrating, we get

$$\int |\nabla u|^2 + |\ell \nabla \theta - q \mathbf{A}|^2 u^2 + W'(u)u = 0.$$

So, since $W'(s)s \geq 0$, we get $u = 0$. ■

3 The existence proof

3.1 The functional setting

Let H^1 denote the usual Sobolev space with norm

$$\|u\|_{H^1}^2 = \int (|\nabla u|^2 + u^2) dx;$$

moreover we need to use also the weighted Sobolev space \hat{H}^1 whose norm is given by

$$\|u\|_{\hat{H}^1}^2 = \int \left[|\nabla u|^2 + \left(1 + \frac{\ell^2}{r^2}\right) u^2 \right] dx, \quad \ell \in \mathbb{Z}$$

where $r = \sqrt{x_1^2 + x_2^2}$. Clearly $\hat{H}^1 = H^1$ when $\ell = 0$.

We set $\mathcal{D} = C_0^\infty(\mathbb{R}^3)$ and we denote by $\mathcal{D}^{1,2}$ the completion of \mathcal{D} with respect to the inner product

$$(v | w)_{\mathcal{D}^{1,2}} = \int \nabla v \cdot \nabla w dx. \quad (72)$$

Here and in the following the dot \cdot will denote the Euclidean inner product in \mathbb{R}^3 .

We set

$$H = \hat{H}^1 \times \mathcal{D}^{1,2} \times (\mathcal{D}^{1,2})^3$$

$$\|(u, \phi, \mathbf{A})\|_H^2 = \int |\nabla u|^2 + \left(1 + \frac{\ell^2}{r^2}\right) u^2 + |\nabla \phi|^2 + |\nabla \mathbf{A}|^2. \quad (73)$$

We shall denote by $u = u(r, x_3)$ the real maps having cylindrical symmetry, i.e. those real maps in \mathbb{R}^3 which depend only from $r = \sqrt{x_1^2 + x_2^2}$ and x_3 . We set

$$\mathcal{D}_r = \{u \in \mathcal{D} : u = u(r, x_3)\} \quad (74)$$

and we shall denote by $\mathcal{D}_r^{1,2}$ (respectively \hat{H}_r^1) the closure of \mathcal{D}_r in the $\mathcal{D}^{1,2}$ (respectively \hat{H}^1) norm.

Now we consider the functional

$$J(u, \phi, \mathbf{A}) = \frac{1}{2} \int |\nabla u|^2 - |\nabla \phi|^2 + |\nabla \times \mathbf{A}|^2$$

$$+ \frac{1}{2} \int \left[|\ell \nabla \theta - q \mathbf{A}|^2 - (\omega - q\phi)^2 \right] u^2 + \int W(u) \quad (75)$$

where $(u, \phi, \mathbf{A}) \in H$. The equations (48), (49) and (50) are the Euler-Lagrange equations of the functional J . Standard computations show that the following lemma holds:

Lemma 10 Assume that W satisfies $W1)$, $W2)$, $W3)$ and

$$\limsup \frac{W'(s)}{s^5} < \infty \text{ for } s \rightarrow \infty. \quad (76)$$

Then the functional J is C^1 on H .

Without loss of generality we can assume that assumption (76) is satisfied. In fact, if (76) is not satisfied, we can replace $W'(s)$ with $W'(\bar{s})s$ for $s > \bar{s}$, where $\bar{s} > 0$ is s.t. $W'(\bar{s}) > 0$. By using a maximum principle argument, it can be seen that, with this nonlinearity, any solution u takes values between 0 and \bar{s} .

By the above lemma it follows that the critical points $(u, \phi, \mathbf{A}) \in H$ of J (with $u \geq 0$) are weak solutions of eq. (48), (49) and (50), namely

$$\int \nabla u \cdot \nabla v + \left[|\ell \nabla \theta - q \mathbf{A}|^2 - (\omega - q\phi)^2 \right] uv + W'(u)v = 0, \quad \forall v \in \hat{H}^1 \quad (77)$$

$$\int \nabla \phi \cdot \nabla w + qu^2 (\omega - q\phi) w = 0, \quad \forall w \in \mathcal{D}^{1,2} \quad (78)$$

$$\int \nabla \mathbf{A} \cdot \nabla \mathbf{V} - qu^2 (\ell \nabla \theta - q \mathbf{A}) \cdot \mathbf{V} = 0, \quad \forall \mathbf{V} \in (\mathcal{D}^{1,2})^3. \quad (79)$$

3.2 Solutions in the sense of distributions

Since \mathcal{D} is not contained in \hat{H}^1 , a solution $(u, \phi, \mathbf{A}) \in H$ of (77), (78), (79) need not to be a solution of (48), (49), (50) in the sense of distributions on \mathbb{R}^3 . In fact, since $\nabla \theta(x)$ is singular on Σ , it might be that for some test function $v \in \mathcal{D}$, when $\ell \neq 0$, the integral $\int |\ell \nabla \theta - q \mathbf{A}|^2 uv$ diverges, unless u is sufficiently small as $x \rightarrow \Sigma$.

In this section we will show that this fact does not occur, namely the singularity is removable in the sense of the following theorem:

Theorem 11 Let $(u_0, \phi_0, \mathbf{A}_0) \in H$, $u_0 \geq 0$ be a solution of (77), (78), (79) (i.e. a critical point of J). Then $(u_0, \phi_0, \mathbf{A}_0)$ is a solution of equations (48), (49) and (50) in the sense of distribution, namely

$$\int \nabla u_0 \cdot \nabla v + \left[|\ell \nabla \theta - q \mathbf{A}_0|^2 - (\omega - q\phi_0)^2 \right] u_0 v + W'(u_0)v = 0, \quad \forall v \in \mathcal{D} \quad (80)$$

$$\int \nabla \phi_0 \cdot \nabla w - qu_0^2 (\omega - q\phi_0) w = 0, \quad \forall w \in \mathcal{D} \quad (81)$$

$$\int \nabla \mathbf{A}_0 \cdot \nabla \mathbf{V} - qu_0^2 (\ell \nabla \theta - q \mathbf{A}_0) \cdot \mathbf{V} = 0, \quad \forall \mathbf{V} \in \mathcal{D}^3. \quad (82)$$

Let χ_n (n positive integer) be a family of smooth functions depending only on $r = \sqrt{x_1^2 + x_2^2}$ and x_3 and which satisfy the following assumptions:

- $\chi_n(r, x_3) = 1$ for $r \geq \frac{2}{n}$
- $\chi_n(r, x_3) = 0$ for $r \leq \frac{1}{n}$
- $|\chi_n(r, x_3)| \leq 1$
- $|\nabla \chi_n(r, x_3)| \leq 2n$
- $\chi_{n+1}(r, x_3) \geq \chi_n(r, x_3)$

Lemma 12 *Let φ be a function in $H^1 \cap L^\infty$ with bounded support and set $\varphi_n = \varphi \cdot \chi_n$. Then, up to a subsequence, we have that*

$$\varphi_n \rightarrow \varphi \text{ weakly in } H^1$$

Proof. Clearly $\varphi_n \rightarrow \varphi$ a.e. Then, by standard arguments, the conclusion holds if we show that $\{\varphi_n\}$ is bounded in H^1 . Clearly $\{\varphi_n\}$ is bounded in L^2 . Let us now prove that

$$\left\{ \int |\nabla \varphi_n|^2 \right\} \text{ is bounded.}$$

We have

$$\begin{aligned} \int |\nabla \varphi_n|^2 &\leq 2 \int |\nabla \varphi \cdot \chi_n|^2 + |\varphi \cdot \nabla \chi_n|^2 \\ &\leq 2 \int |\nabla \varphi|^2 + 2 \int_{\Gamma_\varepsilon} |\varphi \cdot \nabla \chi_n|^2 \end{aligned}$$

where

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^3 : \varphi \neq 0 \text{ and } |\nabla \chi_n(r, z)| \neq 0\}.$$

By our construction, $|\Gamma_\varepsilon| \leq c/n^2$ where c depends only on φ . Thus

$$\begin{aligned} \int |\nabla \varphi_n|^2 &\leq 2 \int |\nabla \varphi|^2 + 2 \|\varphi\|_{L^\infty}^2 \int_{\Gamma_\varepsilon} |\nabla \chi_n|^2 \\ &\leq 2 \int |\nabla \varphi|^2 + 2 \|\varphi\|_{L^\infty}^2 \cdot |\Gamma_\varepsilon| \cdot \|\nabla \chi_n\|_{L^\infty}^2 \\ &\leq 2 \int |\nabla \varphi|^2 + 8c \|\varphi\|_{L^\infty}^2. \end{aligned}$$

Thus φ_n is bounded in H^1 and $\varphi_n \rightarrow \varphi$ weakly in H^1 .

■

Now we are ready to prove Theorem 11

Proof. Clearly (81) and (82) immediately follow by (78) and (79). Let us prove (80). The case $\ell = 0$ is trivial. So assume $\ell \neq 0$. We take any $v \in \mathcal{D}$ and set $\varphi_n = v^+ \chi_n$ where $v^+ = \frac{|v|+v}{2}$. Then, taking φ_n as test function in Eq. (77), we have

$$\int \nabla u_0 \cdot \nabla \varphi_n + \left[|q\mathbf{A}_0 - \ell \nabla \theta|^2 - (q\phi_0 - \omega)^2 \right] u_0 \varphi_n + W'(u_0) \varphi_n = 0 \quad (83)$$

Equation (83) can be written as follows

$$A_n + B_n + C_n + D_n = 0 \quad (84)$$

where

$$A_n = \int \nabla u_0 \cdot \nabla \varphi_n, \quad B_n = \int \left(q^2 \mathbf{A}_0^2 u_0 - (q\phi_0 - \omega)^2 u_0 + W'(u_0) \right) \varphi_n \quad (85)$$

$$C_n = -2 \int q\mathbf{A}_0 \cdot \ell \nabla \theta u_0 \varphi_n, \quad D_n = \int |\ell \nabla \theta|^2 u_0 \varphi_n. \quad (86)$$

By Lemma 12

$$\varphi_n \rightarrow v^+ \text{ weakly in } H^1. \quad (87)$$

Then we have

$$A_n \rightarrow \int \nabla u_0 \cdot \nabla v^+. \quad (88)$$

Now

$$\left(q^2 \mathbf{A}_0^2 u_0 - (q\phi_0 - \omega)^2 u_0 + W'(u_0) \right) \in L^{6/5} = (L^6)'. \quad (89)$$

Then, using again (87) and by the embedding $H^1 \subset L^6$, we have

$$B_n \rightarrow \int \left(q^2 \mathbf{A}_0^2 u_0 - (q\phi_0 - \omega)^2 u_0 + W'(u_0) \right) v^+ < \infty. \quad (89)$$

Now we shall prove that

$$C_n \rightarrow -2 \int q\mathbf{A}_0 \cdot \ell \nabla \theta u_0 v^+ < \infty. \quad (90)$$

Set

$$C = B_R \times [-d, d], \quad B_R = \{(x_1, x_2) \in \mathbb{R}^2 : r^2 = x_1^2 + x_2^2 < R\}$$

where $d, R > 0$ are so large that the cylinder C contains the support of v^+ .

Then

$$\int \left(\frac{\varphi_n}{r}\right)^{\frac{3}{2}} dx = \int_C \left(\frac{v^+ \chi_n}{r}\right)^{\frac{3}{2}} dx \quad (91)$$

$$\leq c_1 \int_{-d}^d \int_0^R \left(\frac{1}{r}\right)^{\frac{3}{2}} r dr dx_3 = M < \infty \quad (92)$$

where $c_1 = 2\pi \sup (v^+)^{\frac{3}{2}}$. By (92) we have

$$\int |\mathbf{A}_0 \cdot \nabla \theta u_0 \varphi_n| dx \leq \|u_0 \mathbf{A}_0\|_{L^3} \left\| \frac{\varphi_n}{r} \right\|_{L^{\frac{3}{2}}} \leq \|u_0 \mathbf{A}_0\|_{L^3} M^{\frac{2}{3}}. \quad (93)$$

Now

$$|\mathbf{A}_0 \cdot \nabla \theta u_0 \varphi_n| \rightarrow |\mathbf{A}_0 \cdot \nabla \theta u_0 v^+| \quad \text{a.e. in } \mathbb{R}^3$$

and the sequence $\{|\mathbf{A}_0 \cdot \nabla \theta u_0 \varphi_n|\}$ is monotone. Then, by the monotone convergence theorem, we get

$$\int |\mathbf{A}_0 \cdot \nabla \theta u_0 \varphi_n| dx \rightarrow \int |\mathbf{A}_0 \cdot \nabla \theta u_0 v^+| dx. \quad (94)$$

By (93) and (94) we deduce that

$$\int |\mathbf{A}_0 \cdot \nabla \theta u_0 v^+| dx < \infty. \quad (95)$$

Then, since

$$|\mathbf{A}_0 \cdot \nabla \theta u_0 \varphi_n| \leq |\mathbf{A}_0 \cdot \nabla \theta u_0 v^+| \in L^1,$$

by the dominated convergence Theorem, we get (90). Finally we prove that

$$D_n \rightarrow \int |\ell \nabla \theta|^2 u_0 v^+ < \infty. \quad (96)$$

By (84), (88), (89) and (90) we have that

$$D_n = \int |\ell \nabla \theta|^2 u_0 \varphi_n \text{ is bounded.} \quad (97)$$

Then the sequence $|\nabla\theta|^2 u_0\varphi_n$ is monotone and it converges a.e. to $|\nabla\theta|^2 u_0v^+$. Then, by the monotone convergence theorem, we get

$$\int |\ell\nabla\theta|^2 u_0\varphi_n dx \rightarrow \int |\ell\nabla\theta|^2 u_0v^+ dx. \quad (98)$$

By (97) and (98) we get (96).

Taking the limit in (84) and by using (88), (89), (90), (96) we have

$$\int \nabla u_0 \cdot \nabla v^+ + \left[|q\mathbf{A}_0 - \ell\nabla\theta|^2 - (q\phi_0 - \omega)^2 \right] u_0v^+ + W'(u_0)v^+ = 0.$$

Taking $\varphi_n = v^-\chi_n$ and arguing in the same way as before, we get

$$\int \nabla u_0 \cdot \nabla v^- + \left[|q\mathbf{A}_0 - \ell\nabla\theta|^2 - (q\phi_0 - \omega)^2 \right] u_0v^- + W'(u_0)v^- = 0.$$

Then

$$\int \nabla u_0 \cdot \nabla v + \left[|q\mathbf{A}_0 - \ell\nabla\theta|^2 - (q\phi_0 - \omega)^2 \right] u_0v + W'(u_0)v = 0.$$

Since $v \in \mathcal{D}$ is arbitrary, we get that equation (80) is solved. ■

The presence of the term $-\int |\nabla\phi|^2$ gives to the functional J a strong indefiniteness, namely any critical point of J has infinite Morse index: this fact is a great obstacle to a direct study of the critical points. To avoid this difficulty we shall introduce a *reduced functional*

3.3 The reduced functional

Equation (49) can be written as follows

$$-\Delta\phi + q^2u^2\phi = q\omega u^2 \quad (99)$$

and it can be easily verified (see [4]) that for any $u \in H^1(\mathbb{R}^3)$, there exists a unique solution $\phi \in \mathcal{D}^{1,2}$ of (99).

Clearly, if $u \in \hat{H}_r^1(\mathbb{R}^3)$, the solution $\phi = \phi_u$ of (99) belongs to $\mathcal{D}_r^{1,2}$. Then we can define the map

$$u \in \hat{H}_r^1(\mathbb{R}^3) \rightarrow Z_\omega(u) = \phi_u \in \mathcal{D}_r^{1,2} \text{ solution of (99)}. \quad (100)$$

Standard arguments show that the map Z_ω is C^1 . Since ϕ_u solves (99), clearly we have

$$d_\phi J(u, Z_\omega(u), \mathbf{A}) = 0 \quad (101)$$

where J is defined in (75) and $d_\phi J$ denotes the partial differential of J with respect to ϕ . For $u \in H^1(\mathbb{R}^3)$, let $\Phi = \Phi_u$ be the solution of the equation (99) with $\omega = 1$, then Φ_u solves the equation

$$-\Delta \Phi_u + q^2 u^2 \Phi_u = q u^2. \quad (102)$$

Clearly

$$\phi_u = \omega \Phi_u. \quad (103)$$

Now let $q > 0$, then by maximum principle arguments it is easy to show that for any $u \in H^1(\mathbb{R}^3)$

$$0 \leq \Phi_u \leq \frac{1}{q}. \quad (104)$$

Now, if $(u, \mathbf{A}) \in \hat{H}^1 \times (\mathcal{D}^{1,2})^3$, we set

$$\tilde{J}(u, \mathbf{A}) = J(u, Z_\omega(u), \mathbf{A})$$

where J is defined in (75). By using the chain rule and equation (101), it is easy to verify (see the first part of the proof of Theorem 16 in ([8])) that

$$\left((u, \mathbf{A}) \text{ critical point of } \tilde{J} \right) \implies ((u, Z_\omega(u), \mathbf{A}) \text{ critical point of } J). \quad (105)$$

We will refer to $\tilde{J}(u, \mathbf{A})$ as the *reduced action functional*.

From (102) we have

$$\int q u^2 \Phi_u dx = \int |\nabla \Phi_u|^2 dx + q^2 \int u^2 \Phi_u^2 dx. \quad (106)$$

Now, by (103), (106), we have:

$$\begin{aligned} \tilde{J}(u, \mathbf{A}) &= J(u, Z_\omega(u), \mathbf{A}) = \frac{1}{2} \int |\nabla u|^2 - |\nabla \phi_u|^2 + |\nabla \times \mathbf{A}|^2 \\ &\quad + \frac{1}{2} \int \left[|\ell \nabla \theta - q \mathbf{A}|^2 - (q \phi_u - \omega)^2 \right] u^2 + \int W(u) \\ &= \frac{1}{2} \int \left(|\nabla u|^2 + |\nabla \times \mathbf{A}|^2 + |\ell \nabla \theta - q \mathbf{A}|^2 u^2 \right) \\ &\quad - \frac{1}{2} \omega^2 \int \left(|\nabla \Phi_u|^2 + q^2 u^2 \Phi_u^2 + u^2 - 2q u^2 \Phi_u \right) + \int W(u) \\ &= \frac{1}{2} \int |\nabla u|^2 + |\nabla \times \mathbf{A}|^2 + |\ell \nabla \theta - q \mathbf{A}|^2 u^2 + \int W(u) \\ &\quad - \frac{\omega^2}{2} \int ([1 - q \Phi_u]) u^2. \end{aligned} \quad (107)$$

Then

$$\tilde{J}(u, \mathbf{A}) = I(u, \mathbf{A}) - \frac{\omega^2}{2} K_q(u) \quad (108)$$

where

$$I(u, \mathbf{A}) = \frac{1}{2} \int |\nabla u|^2 + |\nabla \times \mathbf{A}|^2 + |\ell \nabla \theta - q \mathbf{A}|^2 u^2 + \int W(u)$$

and

$$K_q(u) = \int ([1 - q\Phi_u]) u^2. \quad (109)$$

Now, following the same lines as before, we can define the *reduced energy functional* as follows

$$\tilde{\mathcal{E}}(u, \mathbf{A}) = \mathcal{E}(u, Z_\omega(u), \mathbf{A})$$

Where (see (44))

$$\begin{aligned} \mathcal{E} = & \frac{1}{2} \int \left(|\nabla u|^2 + |\nabla \phi|^2 + |\nabla \times \mathbf{A}|^2 + (|\ell \nabla \theta - q \mathbf{A}|^2 + (\omega - q\phi)^2) u^2 \right) \\ & + \int W(u). \end{aligned} \quad (110)$$

It can be shown as for (108) that

$$\tilde{\mathcal{E}}(u, \mathbf{A}) = I(u, \mathbf{A}) + \frac{\omega^2}{2} K_q(u). \quad (111)$$

Observe that

$$Q = q\sigma = q\omega K_q(u)$$

represents the (electric) charge (see (45) and (46)), so that we can write for $u \neq 0$

$$\tilde{\mathcal{E}}(u, \mathbf{A}) = I(u, \mathbf{A}) + \frac{\omega^2}{2} K_q(u) = I(u, \mathbf{A}) + \frac{\sigma^2}{2K_q(u)}.$$

Then for any $\sigma \neq 0$, the functional defined by

$$E_{\sigma,q}(u, \mathbf{A}) = I(u, \mathbf{A}) + \frac{\sigma^2}{2K_q(u)}, \quad (u, \mathbf{A}) \in \hat{H}^1 \times (\mathcal{D}^{1,2})^3, \quad u \neq 0 \quad (112)$$

represents the energy on the configuration (u, Φ_u, \mathbf{A}) having charge $Q = q\sigma$ or equivalently frequency $\omega = \frac{\sigma}{K_q(u)}$.

The following lemma holds

Lemma 13 Consider the functional

$$\hat{H}^1 \ni u \rightarrow K(u) = \int u^2(1 - q\Phi_u)dx.$$

Then for any $u \in \hat{H}^1$ we have

$$K'(u) = 2u(1 - q\Phi_u)^2. \quad (113)$$

Proof. Set

$$\mathcal{A}(u, \Phi) = \int |\nabla \Phi|^2 dx + \int u^2(1 - q\Phi)^2 dx.$$

By (106) clearly we have

$$\mathcal{A}(u, \Phi_u) = K(u).$$

Then

$$K'(u) = \frac{\partial \mathcal{A}}{\partial u}(u, \Phi_u) + \frac{\partial \mathcal{A}}{\partial \Phi}(u, \Phi_u)\Phi'_u \quad (114)$$

where $\frac{\partial \mathcal{A}}{\partial u}$, $\frac{\partial \mathcal{A}}{\partial \Phi}$ denote the partial derivatives of \mathcal{A} with respect to u and Φ respectively. Since Φ_u solves (102), we have

$$\frac{\partial \mathcal{A}}{\partial \Phi}(u, \Phi_u) = 0.$$

Then (114) gives

$$K'(u) = \frac{\partial \mathcal{A}}{\partial u}(u, \Phi_u) = 2u(1 - q\Phi_u)^2.$$

■

The following proposition holds

Proposition 14 Let $\sigma \neq 0$ and $(u, \mathbf{A}) \in \hat{H}^1 \times (\mathcal{D}^{1,2})^3$, $u \neq 0$ be a critical point of $E_{\sigma,q}$ (see (112)). Then, if we set $\omega = \frac{\sigma}{K_q(u)}$, $(u, Z_\omega(u), \mathbf{A})$ is a critical point of J

Proof. Since $(u, \mathbf{A}) \in \hat{H}^1 \times (\mathcal{D}^{1,2})^3$, $u \neq 0$ is a critical point of $E_{\sigma,q}$, we have

$$0 = E'_{\sigma,q}(u, \mathbf{A}) = I'(u, \mathbf{A}) - \frac{\sigma^2 K'_q(u)}{2K_q(u)^2} = I'(u, \mathbf{A}) - \frac{\omega^2 K'_q(u)}{2}, \quad \omega = \frac{\sigma}{K_q(u)}.$$

Hence (u, \mathbf{A}) is a critical point of the functional

$$\tilde{J}(u, \mathbf{A}) = I(u, \mathbf{A}) - \frac{\omega^2 K_q(u)}{2}.$$

So by (105) $(u, Z_\omega(u), \mathbf{A})$ is a critical point of J . ■

By Proposition 14 and Theorem 11 we are reduced to study the critical points of $E_{\sigma,q}$ which is a functional bounded from below.

However $E_{\sigma,q}$ contains the term $\int |\nabla \times \mathbf{A}|^2$ which is not a Sobolev norm.

In order to avoid this difficulty we introduce a suitable manifold $V \subset \hat{H}^1 \times (\mathcal{D}^{1,2})^3$ such that:

- the critical points of J restricted to V satisfy Eq. (48), (49), 50); namely V is a "natural constraint" for J .
- The components \mathbf{A} of the elements in V are divergence free, then the term $\int |\nabla \times \mathbf{A}|^2$ can be replaced by $\|\mathbf{A}\|_{(\mathcal{D}^{1,2})^3}^2 = \int |\nabla \mathbf{A}|^2$.

We set

$$\mathcal{A}_0 := \{\mathbf{X} \in \mathcal{C}_0^\infty(\mathbb{R}^3 \setminus \Sigma, \mathbb{R}^3) : \mathbf{X} = b(r, x_3) \nabla \theta; b \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma, \mathbb{R})\}. \quad (115)$$

Let \mathcal{A} denote the closure of \mathcal{A}_0 with respect to the norm of $(\mathcal{D}^{1,2})^3$. We shall consider the following space

$$V := \hat{H}_r^1 \times \mathcal{A} \quad (116)$$

where \hat{H}_r^1 is the closure of \mathcal{D}_r with respect to the \hat{H}^1 norm. We shall set $U = (u, \mathbf{A})$ and

$$\|U\|_V = \|(u, \mathbf{A})\|_V = \|u\|_{\hat{H}_r^1} + \|\mathbf{A}\|_{(\mathcal{D}^{1,2})^3}.$$

Lemma 15 *If $\mathbf{A} \in \mathcal{A}$, then*

$$\int |\nabla \times \mathbf{A}|^2 = \int |\nabla \mathbf{A}|^2.$$

Proof. Let $\mathbf{A} = b \nabla \theta \in \mathcal{A}_0$. Since b depends only on r and x_3 , it is easy to check that

$$\nabla b \cdot \nabla \theta = 0.$$

Since θ is harmonic in $\mathbb{R}^3 \setminus \Sigma$ and b has support in $\mathbb{R}^3 \setminus \Sigma$

$$b \Delta \theta = 0.$$

Then

$$\nabla \cdot \mathbf{A} = \nabla \cdot (b \nabla \theta) = \nabla b \cdot \nabla \theta + b \Delta \theta = 0.$$

Thus, by continuity, we get

$$\int (\nabla \cdot \mathbf{A})^2 = 0 \text{ for any } \mathbf{A} \in \mathcal{A}.$$

Then

$$\int |\nabla \times \mathbf{A}|^2 = \int (\nabla \cdot \mathbf{A})^2 + \int |\nabla \times \mathbf{A}|^2 = \int |\nabla \mathbf{A}|^2.$$

■

3.4 Analysis of the minimizing sequences

The ratio energy/charge is a crucial quantity for the following lemmas. For a charge $\sigma > 0$ this ratio is defined as function of u and \mathbf{A} in the following way

$$\Lambda_{\sigma,q}(u, \mathbf{A}) = \frac{E_{\sigma,q}(u, \mathbf{A})}{\sigma} = \frac{I(u, \mathbf{A})}{\sigma} + \frac{\sigma}{2K_q(u)}, \quad (u, \mathbf{A}) \in \hat{H}^1 \times (\mathcal{D}^{1,2})^3, \quad u \neq 0$$

where

$$K_q(u) = \int ([1 - q\Phi_u]) u^2. \quad (117)$$

In the following we shall always assume that the W satisfies W1),W2),W3). Firstst we state the following continuity lemma:

Lemma 16 *Let $u \in H^1$, then*

$$\int (1 - q\Phi_u) u^2 \rightarrow \int u^2 \text{ as } q \rightarrow 0$$

Proof. Clearly it is enough to show that

$$q \int \Phi_u u^2 \rightarrow 0 \text{ as } q \rightarrow 0 \quad (118)$$

Since Φ_u depends on q a little work is needed to prove (118). Since Φ_u solves (102), we have

$$\begin{aligned} \|\Phi_u\|_{\mathcal{D}^{1,2}}^2 + q^2 \int u^2 \Phi_u^2 &= q \int u^2 \Phi_u \leq \\ &\leq q \|u\|_{L^{\frac{12}{5}}}^2 \|\Phi_u\|_{L^6} \end{aligned} \quad (119)$$

and then, if $u \neq 0$, we have

$$\frac{\|\Phi_u\|_{\mathcal{D}^{1,2}}^2}{\|\Phi_u\|_{L^6}} \leq q \|u\|_{L^{\frac{12}{5}}}^2.$$

So, since $\mathcal{D}^{1,2}$ is continuously embedded into L^6 , we easily get

$$\|\Phi_u\|_{\mathcal{D}^{1,2}} \leq c_1 q \|u\|_{L^{\frac{12}{5}}}^2, \quad (120)$$

where c_1 is a positive constant. Then we get

$$q \int u^2 \Phi_u \leq q \|u\|_{L^{\frac{12}{5}}}^2 \|\Phi_u\|_{L^6} \leq c_1 q^2 \|u\|_{L^{\frac{12}{5}}}^4.$$

From which we deduce (118). ■

Lemma 17 *There exist $\sigma, \bar{q} > 0$, such that for all $0 \leq q < \bar{q}$ there exists $u \in \hat{H}_r^1$ such that*

$$\Lambda_{\sigma,q}(u, 0) < 1.$$

Proof. For $0 < \mu < \lambda$ we set:

$$T_{\lambda,\mu} = \{(r, x_3) : (r - \lambda)^2 + x_3^2 \leq \mu\}$$

and, for $\lambda > 2$, we consider a smooth function u_λ with cylindrical symmetry such that

$$u_\lambda(r, x_3) = \begin{cases} s_0 & \text{if } (r, x_3) \in T_{\lambda,\lambda/2} \\ 0 & \text{if } (r, x_3) \notin T_{\lambda,\lambda/2+1} \end{cases}$$

where s_0 is such that $N(s_0) < 0$ (see (61)). Moreover we may assume that

$$|\nabla u_\lambda(r, x_3)| \leq 2 \text{ for } (r, x_3) \in T_{\lambda,\lambda/2+1} \setminus T_{\lambda,\lambda/2}.$$

We have that for all $\sigma \neq 0$

$$\begin{aligned} \Lambda_{\sigma,q}(u_\lambda, 0) &= \frac{1}{\sigma} \int \left[\frac{1}{2} |\nabla u_\lambda|^2 + \frac{\ell^2}{2} \frac{u_\lambda^2}{r^2} + W(u_\lambda) \right] dx + \frac{\sigma}{2K_q(u_\lambda)} \\ &= \frac{\int \left[|\nabla u_\lambda|^2 + \frac{\ell^2 u_\lambda^2}{r^2} \right] dx}{2\sigma} + \frac{\int u_\lambda^2}{2\sigma} + \frac{\int N(u_\lambda) dx}{\sigma} + \frac{\sigma}{2K_q(u_\lambda)}. \end{aligned}$$

Now take

$$\sigma = \sigma_\lambda = \int u_\lambda^2$$

in this case we get

$$\Lambda_{\sigma_\lambda, q}(u_\lambda, 0) = \frac{1}{2} + \frac{\sigma_\lambda}{2K_q(u_\lambda)} + \frac{\int \left[|\nabla u_\lambda|^2 + \frac{\ell^2 u_\lambda^2}{r^2} \right] dx}{2 \int u_\lambda^2} + \frac{\int N(u_\lambda) dx}{\int u_\lambda^2}. \quad (121)$$

By a direct computation we have that

$$\int |\nabla u_\lambda|^2 \leq c_1 \text{meas}(T_{\lambda, \lambda/2+1} \setminus T_{\lambda, \lambda/2}) = c_2 \lambda^2 \quad (122)$$

$$\int \frac{u_\lambda^2}{r^2} \leq \frac{c_3}{\lambda^2} \text{meas}(T_{\lambda, \lambda/2+1}) = c_4 \lambda \quad (123)$$

$$\int u_\lambda^2 \geq c_5 \text{meas}(T_{\lambda, \lambda/2+1}) = c_6 \lambda^3. \quad (124)$$

So that

$$\frac{\int \left[|\nabla u_\lambda|^2 + \frac{\ell^2 u_\lambda^2}{r^2} \right] dx}{2 \int u_\lambda^2} = O\left(\frac{1}{\lambda}\right). \quad (125)$$

Moreover

$$\begin{aligned} \int N(u_\lambda) dx &\leq N(s_0) \text{meas}(T_{\lambda, \lambda/2}) + c_7 \text{meas}(T_{\lambda, \lambda/2+1} \setminus T_{\lambda, \lambda/2}) = \\ &\leq c_8 N(s_0) \lambda^3 + c_9 \lambda^2. \end{aligned} \quad (126)$$

From (126) and (124) we get

$$\frac{\int N(u_\lambda) dx}{\int u_\lambda^2} \leq c_{10} \frac{N(s_0)}{s_0^2} + O\left(\frac{1}{\lambda}\right) = g(s_0, \lambda). \quad (127)$$

From (121), (125) and (127) we get

$$\Lambda_{\sigma_\lambda, q}(u_\lambda, 0) \leq \frac{1}{2} + \frac{\sigma_\lambda}{2K_q(u_\lambda)} + g(s_0, \lambda). \quad (128)$$

Since $N(s_0) < 0$, we can take λ_0 so large that

$$g(s_0, \lambda_0) < 0. \quad (129)$$

Now we take

$$\sigma = \sigma_{\lambda_0} = \int u_{\lambda_0}^2, \text{ and } u = u_{\lambda_0}.$$

Now, by Lemma 16, we have

$$K_q(u) \rightarrow K_0(u) = \sigma \text{ for } q \rightarrow 0.$$

So

$$\frac{\sigma}{2K_q(u)} \rightarrow \frac{1}{2} \text{ for } q \rightarrow 0. \quad (130)$$

Then, by (128), (129) and (130), there is $\bar{q} > 0$ so small that, for all $0 \leq q < \bar{q}$, we have

$$\Lambda_{\sigma,q}(u, 0) \leq \frac{1}{2} + \frac{\sigma}{2K_q(u)} + g(s_0, \lambda_0) < 1.$$

■

Now the following a priori estimate on the minimizing sequences can be obtained

Lemma 18 *Any minimizing sequence $(u_n, \mathbf{A}_n) \subset V$ for $E_{\sigma,q} |_V$ is bounded in $\hat{H}^1 \times (\mathcal{D}^{1,2})^3$.*

Proof. Let $(u_n, \mathbf{A}_n) \subset V$ be a minimizing sequence for $E_{\sigma,q} |_V$. Clearly

$$\|\mathbf{A}_n\|_{(\mathcal{D}^{1,2})^3} \text{ is bounded.}$$

So it remain to prove that

$$\|u_n\|_{\hat{H}_r^1} \text{ is bounded.} \quad (131)$$

To this end we shall first show that

$$\|u_n\|_{L^2} \text{ is bounded.} \quad (132)$$

Since (u_n, \mathbf{A}_n) is a minimizing sequence for $E_{\sigma,q} |_V$ we get

$$\int W(u_n) \text{ and } \int |\nabla u_n|^2 \text{ are bounded.} \quad (133)$$

Then we have also that

$$\int u_n^6 \text{ is bounded.} \quad (134)$$

Let $\varepsilon > 0$ and set

$$\Omega_n = \{x \in \mathbb{R}^3 : |u_n(x)| > \varepsilon\} \text{ and } \Omega_n^c = \mathbb{R}^3 \setminus \Omega_n.$$

By (133) and since $W \geq 0$ we have

$$\int_{\Omega_n^c} W(u_n) \text{ is bounded.} \quad (135)$$

By W_2) we can write

$$W(s) = \frac{1}{2}s^2 + o(s^2).$$

Then, if ε is small enough, there is a constant $c > 0$ such that

$$\int_{\Omega_n^c} W(u_n) \geq c \int_{\Omega_n^c} u_n^2. \quad (136)$$

By (135) and (136) we get that

$$\int_{\Omega_n^c} u_n^2 \text{ is bounded.} \quad (137)$$

On the other hand

$$\int_{\Omega_n} u_n^2 \leq \left(\int_{\Omega_n} u_n^6 \right)^{\frac{1}{3}} \text{meas}(\Omega_n)^{\frac{2}{3}}. \quad (138)$$

By (134) we have that

$$\text{meas}(\Omega_n) \text{ is bounded.} \quad (139)$$

By (138), (139) and again by (134) we get

$$\int_{\Omega_n} u_n^2 \text{ is bounded.} \quad (140)$$

So (132) follows from (137) and (140).

Let us finally prove (131).

Clearly

$$\begin{aligned} E_{\sigma,q}(u_n, \mathbf{A}_n) &\geq I(u_n, \mathbf{A}_n) \geq \\ &\frac{1}{2} \int \left(|\nabla u_n|^2 + |\nabla \mathbf{A}_n|^2 + q^2 |\mathbf{A}_n|^2 u_n^2 + \ell^2 \frac{u_n^2}{r^2} - 2q \frac{\ell}{r} |\mathbf{A}_n| |u_n|^2 \right) dx \geq \\ &\frac{1}{2} \|u_n\|_{\dot{H}_r^1}^2 - q \int \frac{\ell}{r} |\mathbf{A}_n| |u_n|^2 - \sup \|u_n\|_{L^2} \end{aligned} \quad (141)$$

Also we have

$$\begin{aligned} \int \frac{q\ell}{r} |\mathbf{A}_n| |u_n|^2 &\leq \frac{1}{2} \int \left(4q^2 \ell^2 |\mathbf{A}_n|^2 + \frac{1}{4r^2} \right) |u_n|^2 \leq \\ &\frac{1}{8} \|u_n\|_{\dot{H}_r^1}^2 + 2q^2 \ell^2 \int |\mathbf{A}_n|^2 |u_n|^2. \end{aligned} \quad (142)$$

Since $E_{\sigma,q}(u_n, \mathbf{A}_n)$ is bounded, by (141) and (142) we deduce that

$$c_1 \geq \left(\frac{1}{2} - \frac{1}{8} \right) \|u_n\|_{\dot{H}_r^1}^2 - 2q^2 \ell^2 \int |\mathbf{A}_n|^2 |u_n|^2. \quad (143)$$

Here c_1, c_2 will denote suitable constants.

Now, since $\|u_n\|_{L^2}$ and $\|u_n\|_{L^6}$ are bounded, also $\|u_n\|_{L^3}$ is bounded.

Then, by using also the boundeness of $\|\mathbf{A}_n\|_{L^6}$, we get

$$\int |\mathbf{A}_n|^2 |u_n|^2 \leq (\|\mathbf{A}_n\|_{L^6})^{\frac{1}{3}} (\|u_n\|_{L^3})^{\frac{2}{3}} \leq c_2. \quad (144)$$

From (143) and (144) we deduce the boundeness of $\|u_n\|_{\dot{H}_r^1}^2$. ■

By Lemma 18 any minimizing sequence $U_n := (u_n, \mathbf{A}_n) \subset V$ of $E_{\sigma,q}|_V$ weakly converges (up to a subsequence). Observe that $E_{\sigma,q}$ is invariant for translations along the x_3 -axis, namely for $U \in V$ and $L \in \mathbb{R}$ we have

$$E_{\sigma,q}(T_L U) = E_{\sigma,q}(U)$$

where

$$T_L(U)(x_1, x_2, x_3) = U(x_1, x_2, x_3 + L). \quad (145)$$

As consequence of this invariance we have that (u_n, \mathbf{A}_n) does not contain in general a (strongly) convergent subsequence. So we argue as follows: we prove that for suitable σ, q there exists a minimizing sequence (u_n, \mathbf{A}_n) of $E_{\sigma,q}|_V$ which, up to translations along the x_3 -direction, weakly converges to a non trivial limit (u_0, \mathbf{A}_0) . This limit will be actually a critical point of $E_{\sigma_0,q}$ for some charge σ_0 .

To follow the above program we first prove the following Lemma

Lemma 19 *Let $U_n = (u_n, \mathbf{A}_n) \subset V$ be a minimizing sequence of $E_{\sigma,q}|_V$, $\sigma > 0$. Then there exist $\delta, M > 0$ such that*

$$\delta \leq \omega_n \leq M$$

where

$$\omega_n = \frac{\sigma}{K_q(u_n)}.$$

Proof. Since $(u_n, \mathbf{A}_n) \subset V$ is a minimizing sequence of the functional $E_{\sigma,q} \mid_V$ defined by

$$E_{\sigma,q}(u, \mathbf{A}) = I(u, \mathbf{A}) + \frac{\sigma^2}{2K_q(u)},$$

we have that for some constant $c_1 > 0$

$$c_1 \leq K_q(u_n). \quad (146)$$

Also for some constant $c_2 > 0$ we have

$$K_q(u_n) \leq c_2. \quad (147)$$

In fact, arguing by contradiction, we assume that, up to a subsequence

$$K_q(u_n) = \int ([1 - q\Phi_{u_n}]) u_n^2 \rightarrow \infty,$$

then by (104) also we get

$$\int u_n^2 \rightarrow \infty$$

contradicting (132).

Finally the conclusion immediately follows from (146) and (147). ■

Now we shall prove the following proposition

Proposition 20 *There exist $\sigma, \bar{q} > 0$ such that for all $0 \leq q < \bar{q}$, for any minimizing sequence $(u_n, \mathbf{A}_n) \subset V$ of $E_{\sigma,q} \mid_V$ we have*

$$\|u_n\|_{L^3} \geq c > 0 \text{ for } n \text{ large.}$$

Proof. Let σ and q be chosen as required in Lemma 17. Now let $(u_n, \mathbf{A}_n) \subset V$ be a minimizing sequence of $E_{\sigma,q}$ and hence of $\Lambda_{\sigma,q}$. Then by Lemma 17 we get

$$\Lambda_{\sigma,q}(u_n, \mathbf{A}_n) \leq 1 - \delta, \quad \delta > 0 \quad (148)$$

Then we have also

$$\frac{\int \left[|\nabla u_n|^2 + \frac{\ell^2 u_n^2}{r^2} \right] dx}{2\sigma} + \frac{\int u_n^2}{2\sigma} + \frac{\int N(u_n) dx}{\sigma} + \frac{\sigma}{2 \int u_n^2} \leq 1 - \delta.$$

Thus

$$\frac{\int N(u_n)dx}{\sigma} \leq 1 - \delta - \left(\frac{\int u_n^2}{2\sigma} + \frac{\sigma}{2 \int u_n^2} \right) \leq -\delta.$$

This implies that

$$\int N(u_n)dx \leq -\delta\sigma.$$

On the other hand, by the assumptions on W , we have that

$$N(s) \geq -bs^3, \quad b > 0$$

Then

$$b \int |u_n|^3 \geq - \int N(u_n)dx \geq \delta\sigma.$$

■

Proposition 21 *For any $\sigma, q \geq 0$ there exists a minimizing sequence (u_n, \mathbf{A}_n) of $E_{\sigma,q} \mid_V$, with $u_n \geq 0$ and which is also a P.S. sequence for $E_{\sigma,q}$, i.e.*

$$E'_{\sigma,q}(u_n, \mathbf{A}_n) \rightarrow 0.$$

Proof. Let $(u_n, \mathbf{A}_n) \subset V$ be a minimizing sequence for $E_\sigma \mid_V$. It is not restrictive to assume that $u_n \geq 0$, in fact, if not, we can replace u_n with $|u_n|$ (see (110)). By standard variational arguments we can also assume that (u_n, \mathbf{A}_n) is a P.S. sequence for $E_\sigma \mid_V$, namely we can assume that

$$E'_{\sigma,q} \mid_V (u_n, \mathbf{A}_n) \rightarrow 0.$$

By using the same arguments in proving Theorem 16 in [8], it can be shown that (u_n, \mathbf{A}_n) is a P.S. sequence also for $E_{\sigma,q}$, i.e.

$$E'_{\sigma,q}(u_n, \mathbf{A}_n) \rightarrow 0. \tag{149}$$

■

Proposition 22 *There exist $\sigma, \bar{q} > 0$ such that for all $0 \leq q < \bar{q}$ there exists a P.S. sequence $U_n = (u_n, \mathbf{A}_n)$ for $E_{\sigma,q}$ which weakly converges to (u_0, \mathbf{A}_0) , $u_0 \geq 0$ and $u_0 \neq 0$.*

Proof. Take σ, q as in Proposition 20. By Proposition 21 there exists a minimizing sequence $U_n = (u_n, \mathbf{A}_n)$ of $E_{\sigma,q} |_V$ with $u_n \geq 0$ and which is also a P.S. sequence for $E_{\sigma,q}$, i.e.

$$E'_{\sigma,q}(U_n) \rightarrow 0.$$

By Proposition 20 we can assume that

$$\|u_n\|_{L^3} \geq c > 0 \text{ for } n \text{ large.}$$

By Lemma 18 the sequence $\{U_n\}$ is bounded in $\hat{H}^1 \times (\mathcal{D}^{1,2})^3$ so we can assume that it weakly converges. However the weak limit could be trivial. We will show that there is a sequence of integers j_n such that (up to a subsequence) $V_n := T_{j_n} U_n \rightharpoonup U_0 = (u_0, \mathbf{A}_0)$, $u_0 \neq 0$, weakly in $H^1 \times (\mathcal{D}^{1,2})^3$.

We set

$$\Omega_j = \{(x_1, x_2, x_3) : j \leq x_3 < j+1\}, j \text{ integer}$$

In the following c_1, \dots, c_4 denote positive constants.

We have for n large

$$\begin{aligned} 0 < c_1 &\leq \|u_n\|_{L^3}^3 = \sum_j \int_{\Omega_j} |u_n|^3 = \sum_j \left(\int_{\Omega_j} |u_n|^3 \right)^{1/3} \cdot \left(\int_{\Omega_j} |u_n|^3 \right)^{2/3} \\ &\leq \sup_j \|u_n\|_{L^3(\Omega_j)} \sum_j \left(\int_{\Omega_j} |u_n|^3 \right)^{2/3} \leq c_2 \cdot \sup_j \|u_n\|_{L^3(\Omega_j)} \cdot \sum_j \|u_n\|_{H^1(\Omega_j)}^2 \\ &\leq c_2 \cdot \sup_j \|u_n\|_{L^3(\Omega_j)} \cdot \|u_n\|_{H^1(\mathbb{R}^3)}^2 \leq (\text{since } \|u_n\|_{H^1(\mathbb{R}^3)}^2 \leq c_3) \\ &\leq c_2 c_3 \sup_j \|u_n\|_{L^3(\Omega_j)}. \end{aligned}$$

Then, for n large, there exists an integer j_n such that

$$\|u_n\|_{L^3(\Omega_{j_n})} \geq \frac{c_1}{2c_2c_3} := c_4 > 0. \quad (150)$$

Now set

$$(u'_n, \mathbf{A}'_n) = U'_n(x_1, x_2, x_3) = U_n(x_1, x_2, x_3 + j_n) = T_{j_n}(U_n)$$

By Lemma 18 the sequence u'_n is bounded in $\hat{H}^1(\mathbb{R}^3)$, then (up to a subsequence) it converges weakly to $u_0 \in \hat{H}^1(\mathbb{R}^3)$. Clearly $u_0 \geq 0$, since $u'_n \geq 0$. We want to show that $u_0 \neq 0$. Now, let $\varphi = \varphi(x_3)$ be a nonnegative, C^∞ -function whose value is 1 for $0 < x_3 < 1$ and 0 for $|x_3| > 2$. Then, the

sequence $\varphi u'_n$ is bounded in $H_0^1(\mathbb{R}^2 \times (-2, 2))$, moreover $\varphi u'_n$ has cylindrical symmetry. Then, using the compactness result proved in [18], we have

$$\varphi u'_n \rightarrow \chi \text{ strongly in } L^3(\mathbb{R}^2 \times (-2, 2)).$$

On the other hand

$$\varphi u'_n \rightarrow \varphi u_0 \text{ a.e.} \quad (151)$$

Then

$$\varphi u'_n \rightarrow \varphi u_0 \text{ strongly in } L^3(\mathbb{R}^2 \times (-2, 2)). \quad (152)$$

Moreover by (150)

$$\|\varphi u'_n\|_{L^3(\mathbb{R}^2 \times (-2, 2))} \geq \|u'_n\|_{L^3(\Omega_0)} = \|u_n\|_{L^3(\Omega_{j_n})} \geq c_4. \quad (153)$$

Then by (152) and (153)

$$\|\varphi u_0\|_{L^3(\mathbb{R}^2 \times (-2, 2))} \geq c_4 > 0.$$

Thus we have that $u_0 \neq 0$. ■

Proposition 23 *There exists $\bar{q} > 0$ such that, for all $0 \leq q < \bar{q}$, for some charge $\sigma_0 > 0$, $E_{\sigma_0, q}$ has a critical point (u_0, \mathbf{A}_0) $u_0 \neq 0$, $u_0 \geq 0$.*

Proof. Let $\sigma, q > 0$ be as in Proposition 22, then there exists a sequence $U_n = (u_n, \mathbf{A}_n)$ in V , with $u_n \geq 0$ and such that

$$E'_{\sigma, q}(u_n, \mathbf{A}_n) \rightarrow 0 \quad (154)$$

and

$$(u_n, \mathbf{A}_n) \rightarrow (u_0, \mathbf{A}_0) \text{ weakly, } u_0 \neq 0$$

Since $u_n \geq 0$ we have $u_0 \geq 0$.

Let us show that $U_0 = (u_0, \mathbf{A}_0)$ is a critical point of $E_{\sigma_0, q}$ for some charge $\sigma_0 > 0$.

By (154) we get that

$$dE_{\sigma, q}(U_n)[w, 0] \rightarrow 0 \text{ and } dE_{\sigma, q}(U_n)[0, \mathbf{w}] \rightarrow 0 \text{ for any } (w, \mathbf{w}) \in \hat{H}^1 \times (\mathcal{D}^{1,2})^3.$$

Then for any $w \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma)$ and $\mathbf{w} \in (C_0^\infty(\mathbb{R}^3))^3$ we have

$$d_u I(U_n)[w] + d_u \left(\frac{\sigma^2}{2K_q(u_n)} \right) [w] \rightarrow 0 \quad (155)$$

and

$$d_{\mathbf{A}} I(U_n) [\mathbf{w}] \rightarrow 0 \quad (156)$$

where d_u and $d_{\mathbf{A}}$ denote the partial differentials of I with respect u and \mathbf{A} . So from (155) we get for any $w \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma)$

$$d_u I(U_n) [w] - \frac{\sigma^2 K'_q(u_n)}{2 (K_q(u_n))^2} [w] \rightarrow 0$$

which can be written as follows

$$d_u I(U_n) [w] - \frac{\omega_n^2 K'_q(u_n)}{2} [w] \rightarrow 0 \quad (157)$$

where

$$\omega_n = \frac{\sigma}{K_q(u_n)}.$$

By Lemma 19 we have (up to a subsequence)

$$\omega_n \rightarrow \omega_0 > 0$$

Then by (157) we get for any $w \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma)$

$$d_u I(U_n) [w] - \frac{\omega_0^2 K'_q(u_n)}{2} [w] \rightarrow 0. \quad (158)$$

Now let Φ_n be the solution in $\mathcal{D}^{1,2}$ of the equation

$$-\Delta \Phi_n + q^2 u_n^2 \Phi_n = q u_n^2. \quad (159)$$

Since $\{u_n\}$ is bounded in H^1 (see (131) and (132)) and since Φ_n solves (159), standard Sobolev estimates show that $\{\Phi_n\}$ is bounded in $\mathcal{D}^{1,2}$ and that its weak limit (up to subsequence) Φ_0 is a weak solution of

$$-\Delta \Phi_0 + q^2 u_0^2 \Phi_0 = q u_0^2. \quad (160)$$

Then, by Lemma 13, we have

$$K'_q(u_n) = 2u_n(1 - q\Phi_n)^2 \text{ and } K'_q(u_0) = 2u_0(1 - q\Phi_0)^2. \quad (161)$$

By standard calculations we have:

$$\begin{aligned} & \text{for any } w \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma) \\ & \int u_n(1 - q\Phi_n)^2 w \rightarrow \int u_0(1 - q\Phi_0)^2 w. \end{aligned} \quad (162)$$

Then, by (161) and (162), we get for any $w \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma)$

$$K'_q(u_n)[w] \rightarrow K'_q(u_0)[w]. \quad (163)$$

Similar standard estimates show that for any $w \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma)$

$$d_u I(U_n)[w] \rightarrow d_u I(U_0)[w]. \quad (164)$$

Then, passing to the limit in (158), by (163) and (164), we get

$$d_u I(U_0)[w] - \frac{\omega_0^2 K'_q(u_0)}{2}[w] = 0 \text{ for any } w \in C_0^\infty(\mathbb{R}^3 \setminus \Sigma). \quad (165)$$

On the other hand similar arguments show that we can pass to the limit also in $d_{\mathbf{A}} I(U_n)[\mathbf{w}]$ and have

$$\begin{aligned} & \text{for all } \mathbf{w} \in (C_0^\infty(\mathbb{R}^3))^3 \\ d_{\mathbf{A}} I(U_n)[\mathbf{w}] & \rightarrow d_{\mathbf{A}} I(U_0)[\mathbf{w}]. \end{aligned} \quad (166)$$

From (156) and (166) we get

$$d_{\mathbf{A}} I(U_0)[\mathbf{w}] = 0 \text{ for all } \mathbf{w} \in (C_0^\infty(\mathbb{R}^3))^3. \quad (167)$$

By (165) and (167) we deduce, by using density and continuity arguments, that $U_0 = (u_0, \mathbf{A}_0)$ is a critical point of $E_{\sigma_0, q}$ with $\sigma_0 = \omega_0 K_q(u_0) > 0$. ■

Proof of Theorem 3

Proof. The first part of Theorem 3 immediately follows from Propositions 23, 14 and Theorem 11. In fact, if u_0, \mathbf{A}_0 are like in Proposition 23, by Proposition 14 and Theorem 11 we deduce that $(u_0, \omega_0, \phi_0, \mathbf{A}_0)$ with $\omega_0 = \frac{\sigma_0}{K_q(u_0)}$, $\phi_0 = Z_{\omega_0}(u_0)$ solves (48), (49), (50).

Now assume $q = 0$, then, by (49) and (50), we easily deduce that $\phi_0 = 0$ and $\mathbf{A}_0 = 0$. Finally assume that $q > 0$. Then, since $\omega_0 > 0$, by (49) we deduce that $\phi_0 \neq 0$. Moreover by (50) we easily deduce that $\mathbf{A}_0 \neq 0$ if and only if $\ell \neq 0$. ■

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